

# Variational Principle for Slow Viscous Flow of Fluids with Anisotropic and Spatially Inhomogeneous Viscosity, Upper and Lower Bounds on the Frictional Force and Torque

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The frictional force and torque exerted by a fluid on a solid body which performs a slow translational or rotational motion is related to the rate of entropy production. Starting from the Navier-Stokes equation, a variational principle is derived for this quantity. Provided that the antisymmetric part of the viscosity tensor is smaller than its symmetric part upper and lower bounds on the force and torque are established. These bounds can be calculated with certain trial functions for the flow velocity field.

The calculation of the frictional force or the frictional torque exerted by a fluid on a solid particle which performs a (slow) uniform translational or rotational motion, in principle, requires the determination of the flow velocity from the steady state (linearized) Navier-Stokes equation. The solution of this equation, however, may be a rather difficult task for fluids with a spatially inhomogeneous or an anisotropic viscosity. The existence of upper and lower bounds on the force and torque which can be calculated with the help of certain trial velocity functions are of great practical help for problems of this type. For spatially homogeneous and isotropic fluids, the existence of an upper bound is well-known (principle of minimum entropy production in the stationary state) and the existence of a lower bound has also been established quite some time ago. It is the purpose of this article to derive similar relations for inhomogeneous and anisotropic fluids. This extension is straightforward and nothing really new for the case where the viscosity tensor is symmetric. For a nonsymmetric viscosity tensor (e.g. fluid in the presence of a magnetic field), however, some modifications have to be made.

This article proceeds as follows. Firstly, the Navier-Stokes equation and the resulting energy balance equation are stated. Then it is indicated that the rate of heat generated by the viscous flow (energy dissipation) is proportional to the component of the frictional force (torque) which is

antiparallel to the velocity (angular velocity) of a particle which undergoes a translational (rotational) motion in the fluid. The remainder of the article is devoted to the derivation a variational principle and of upper and lower bounds on the energy dissipation. Of crucial importance is a functional (scalar product) which is bilinear in the gradient of divergence-free (trial) velocity functions and which involves the spatially dependent viscosity tensor. Some general properties of this scalar product are discussed. The variational principle (extremal property of the rate of entropy production) is proved for the most general case. The bounds on the force and torque can be established only if the antisymmetric part of the viscosity tensor is smaller than its symmetric part. For the calculation of the upper bound, trial functions are needed which just fulfill the appropriate boundary conditions. For the lower bound, in addition, trial functions are required which obey the Navier-Stokes equation but need not fulfill the boundary conditions.

## Navier-Stokes Equation, Energy Balance

The velocity field  $\mathbf{v}$  of a viscous fluid is governed by the Navier-Stokes equation

$$\rho \frac{\partial v_\mu}{\partial t} + \rho v_\nu \nabla_\nu v_\mu + \nabla_\mu p + \nabla_\nu \overline{p_{\nu\mu}} = 0. \quad (1)$$

For an incompressible flow to be considered in the following one has

$$\nabla_\mu v_\mu = 0 \quad (2)$$

and the mass density  $\rho$  is constant. In (1),  $p$  is the hydrostatic pressure. The (symmetric traceless) friction pressure tensor is related to the gradient of  $\mathbf{v}$  by

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$$\overline{p_{\mu\nu}} = -2\eta_{\mu\nu, \mu'r'} \overline{\nabla_{\mu'} v_{r'}} \quad (3)$$

where  $\eta_{\dots}$  is the shear viscosity tensor. The symbol  $\overline{\dots}$  denotes the symmetric traceless part of a tensor, i.e.

$$\overline{a_{\mu\nu}} = \frac{1}{2}(a_{\mu\nu} + a_{\nu\mu}) - \frac{1}{3}a_{\lambda\lambda} \delta_{\mu\nu}$$

for any 2nd rank tensor  $a_{\mu\nu}$ . In a spatially inhomogeneous fluid  $\eta_{\dots}$  depends on the position vector  $\mathbf{r}$ . The tensorial character of  $\eta_{\dots}$  is associated with the anisotropy of the fluid which may e.g. be caused by the presence of external electric or magnetic fields or by the intrinsic anisotropy of a liquid crystal. For an isotropic fluid, (3) reduces to  $\overline{p_{\mu\nu}} = -2\eta \overline{\nabla_{\mu} v_{\nu}}$  with the scalar viscosity coefficient  $\eta = \eta(\mathbf{r})$ . Non-Newtonian effects are disregarded, i.e. it is assumed that the viscosity tensor is independent of the velocity gradient.

Multiplication of Eq. (1) by  $v_{\mu}$ , integration over the volume of the fluid and use of the Gauss theorem yields the energy balance equation

$$\begin{aligned} \frac{d}{dt} \int \frac{1}{2} \rho v^2 d^3r + (\mathbf{v}, \mathbf{v}) \\ = - \int v_{\mu} n_r [\overline{p_{\mu\nu}} + (p + \frac{1}{2} \rho v^2) \delta_{\mu\nu}] d\sigma \end{aligned} \quad (4)$$

where  $\mathbf{n}$  is a unit vector which is normal to the surface of the fluid and is pointing outward from it;  $\int \dots d\sigma$  denotes the integration over the surface of the fluid. The scalar product (which will play a key role in the formulation of the variational principle) is defined by

$$(\mathbf{u}, \mathbf{w}) = 2 \int \overline{\nabla_{\mu} u_r} \eta_{\mu\nu, \mu'r'} \overline{\nabla_{\mu'} w_{r'}} d^3r \quad (5)$$

where it is understood that the integration is over the volume of the fluid. Notice that  $(\mathbf{v}, \mathbf{v}) > 0$  occurring in (4) is the heat generated per unit time by the viscous flow, i.e. it is proportional to the rate of entropy production.

### Force and Torque

The following considerations are restricted to the case of a stationary and slow viscous flow. Instead of (1), the equation

$$\nabla_{\mu} p + \nabla_r \overline{p_{r\mu}} = 0 \quad (6)$$

is used with (2) and (3). Then (4) reduces to

$$(\mathbf{v}, \mathbf{v}) = - \int v_{\mu} n_r (p \delta_{\mu\nu} + \overline{p_{\mu\nu}}) d\sigma. \quad (7)$$

Now  $(\mathbf{v}, \mathbf{v})$  can be related to the force  $\mathbf{F}$  and the torque  $\mathbf{T}$  which the fluid exerts on a solid body

undergoing a translational or a rotational motion in the fluid.

Firstly, the case of a translational motion (Stokes' problem) is considered where the solid body is at rest and the fluid flows with the constant velocity  $-\mathbf{V}$  far away from the obstacle (or equivalently, the solid body moved with constant velocity  $\mathbf{V}$  and the fluid is at rest far away from it). Subject to the restrictions mentioned above and if no-slip boundary conditions are assumed, Eq. (7) yields

$$(\mathbf{v}, \mathbf{v}) = -\mathbf{V} \cdot \mathbf{F}. \quad (8)$$

For a solid body which rotates with constant angular velocity  $\mathbf{\Omega}$ , similarly

$$(\mathbf{v}, \mathbf{v}) = -\mathbf{\Omega} \cdot \mathbf{T} \quad (9)$$

is obtained from (7).

Expression analogous to (8) and (9) apply to plane and cylindrical Couette flow.

According to (8) and (9), bounds on  $(\mathbf{v}, \mathbf{v})$  to be derived in the following provide bounds on the components of the force  $\mathbf{F}$  and the torque  $\mathbf{T}$  which are (anti-)parallel to  $\mathbf{V}$  and  $\mathbf{\Omega}$ , respectively.

### General Properties of the Scalar Product

Before the variational principle is stated, some properties of the scalar product as defined by (5) have to be discussed. Provided that the shear viscosity tensor is symmetric ( $\eta_{\mu\nu, \mu'r'} = \eta_{\mu'r', \mu\nu}$ ), one has  $(\mathbf{u}, \mathbf{w}) = (\mathbf{w}, \mathbf{u})$  and the derivation of upper and lower bounds on  $(\mathbf{v}, \mathbf{v})$  then is formally equivalent to a similar problem encountered in the mechanics of elastic solids [1, 2]; for the treatment of closely related problems cf. Refs. [3, 4]. The shear viscosity tensor is indeed symmetric for a fluid in the presence of an external electric field, in a nematic liquid crystal and, a fortiori, for the case where the viscosity tensor is isotropic. In general, however, the viscosity tensor is not symmetric and consequently  $(\mathbf{u}, \mathbf{w})$  is not symmetric against the exchange of  $\mathbf{u}$  and  $\mathbf{w}$ . In particular, in the presence of a magnetic field  $\mathbf{H}$  one has [5, 6]

$$\eta_{\mu\nu, \mu'r'}(\mathbf{H}) = \eta_{\mu'r', \mu\nu}(-\mathbf{H}). \quad (10)$$

For this case, some general properties of the scalar product are stated and proved.

#### i) Symmetry

Let the tilde " $\sim$ " denote the symmetry operation  $\mathbf{H} \rightarrow -\mathbf{H}$ , i.e. reversal of the sign of the magnetic

field; in particular

$$\tilde{\mathbf{v}}(\mathbf{H}) = \mathbf{v}(-\mathbf{H}). \quad (11)$$

Now one has, per definition,

$$(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) = 2 \int \overline{\nabla_\mu u_\nu(-\mathbf{H})} \eta_{\mu\nu, \mu'\nu'}(-\mathbf{H}) \cdot \overline{\nabla_{\mu'} w_{\nu'}(-\mathbf{H})} d^3r.$$

Use of the symmetry (10) and of the definition (11) then leads to the generalized symmetry relation

$$(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) = (\tilde{\mathbf{w}}, \tilde{\mathbf{u}}). \quad (12)$$

With  $\mathbf{u} = \mathbf{v}(-\mathbf{H}) = \tilde{\mathbf{v}}$ ,  $\mathbf{w} = \mathbf{v}$ ,

$$(\tilde{\tilde{\mathbf{v}}}, \mathbf{v}) = (\tilde{\mathbf{v}}, \mathbf{v}) \quad (13)$$

is inferred from (12). This implies that the scalar product  $(\tilde{\mathbf{v}}, \mathbf{v})$  is an even function in  $\mathbf{H}$ .

### ii) Orthogonality Relations

Let  $\mathbf{v}'$  and  $\mathbf{v}''$  be velocity functions with the properties:

$\mathbf{v}'$  obeys the appropriate boundary conditions and is divergence free,

$\mathbf{v}''$  obeys the differential equation (6) with (3).

Notice that the true solution  $\mathbf{v}$  obeys these two requirements simultaneously. The difference between two distinct functions of the  $\mathbf{v}'$ -type is denoted by  $\delta\mathbf{v}'$ . Clearly,

$\delta\mathbf{v}'$  vanishes at the boundaries of the fluid.

The function  $\delta\mathbf{v}'$  is orthogonal to  $\mathbf{v}''$  in the sense that

$$(\delta\mathbf{v}', \mathbf{v}'') = 0. \quad (14)$$

To prove this relation, it is noticed that integration by parts yields

$$(\delta\mathbf{v}', \mathbf{v}'') = 2 \int n_\nu \delta v_\mu' \eta_{\mu\nu, \mu'\nu'} \overline{\nabla_{\mu'} v_{\nu'}''} d\sigma - \int \delta v_\mu' \nabla_\nu (2\eta_{\mu\nu, \mu'\nu'} \overline{\nabla_{\mu'} v_{\nu'}''}) d^3r.$$

The surface integral vanishes because  $\delta\mathbf{v}'$  vanishes at the boundaries of the fluid. The remaining volume integral can be rewritten as  $-\int \delta v_\mu' \nabla_\mu p d^3r$  since  $\mathbf{v}''$  is assumed to obey Eq. (6) with (3). Again, integration by parts yields a surface integral (which vanishes for the same reason as before) and the volume integral  $\int p \nabla_\mu \delta v_\mu' d^3r$  which vanishes because  $\delta\mathbf{v}'$  is divergence free. Similarly, the adjoint orthogonality relation

$$(\tilde{\mathbf{v}}'', \delta\mathbf{v}') = 0 \quad (15)$$

can be derived. Notice that  $\tilde{\mathbf{v}}''(\mathbf{H}) = \mathbf{v}''(-\mathbf{H})$  is a solution of Eq. (6) where  $\eta_{\mu\nu, \mu'\nu'}$  is replaced by  $\eta_{\mu'\nu', \mu\nu}$ . It is assumed that the boundary conditions are not affected by the reversal of the magnetic field. More specifically, one has  $\mathbf{v}' = \tilde{\mathbf{v}}'$  at the boundaries. Due to Eq. (14), this implies

$$(\mathbf{v}', \mathbf{v}'') = (\tilde{\mathbf{v}}', \mathbf{v}'') \quad (16)$$

for velocity functions of the  $\mathbf{v}'$  and  $\mathbf{v}''$  type as defined above.

### iii) Metric

For the derivation of the variational principle, a metric based on the scalar product of the form  $(\tilde{\mathbf{u}}, \mathbf{w})$  rather than  $(\mathbf{u}, \mathbf{w})$  has to be used where  $\mathbf{u}$  and  $\mathbf{w}$  are (divergence free) velocity fields. The existence of upper and lower bounds on the force and torque can be established only if the expression  $(\tilde{\mathbf{u}}, \mathbf{u})$  is positive. For a symmetric viscosity tensor, where  $\tilde{\mathbf{u}} \equiv \mathbf{u}$  and  $(\tilde{\mathbf{u}}, \mathbf{u}) \equiv (\mathbf{u}, \mathbf{u}) > 0$  this condition is always fulfilled. In the case of a nonsymmetric viscosity tensor,  $(\tilde{\mathbf{u}}, \mathbf{u})$  is positive definite provided that the antisymmetric part

$$\eta_{\mu\nu, \mu'\nu'}^{(s)} = \frac{1}{2} (\eta_{\mu\nu, \mu'\nu'} - \eta_{\mu'\nu', \mu\nu}) \quad (17)$$

of  $\eta_{\dots}$  is not too large compared with its symmetric part

$$\eta_{\mu\nu, \mu'\nu'}^{(s)} = \frac{1}{2} (\eta_{\mu\nu, \mu'\nu'} + \eta_{\mu'\nu', \mu\nu}). \quad (18)$$

This can be inferred from the expression

$$(\tilde{\mathbf{u}}, \mathbf{u}) = 2 \int d^3r [\overline{\nabla_\mu u_\nu^s} \eta_{\mu\nu, \mu'\nu'}^{(s)} \overline{\nabla_{\mu'} u_\nu^s} + 2 \overline{\nabla_\mu u_\nu^s} \eta_{\mu\nu, \mu'\nu'}^{(a)} \overline{\nabla_{\mu'} u_\nu^a} - \overline{\nabla_\mu u_\nu^a} \eta_{\mu\nu, \mu'\nu'}^{(s)} \overline{\nabla_{\mu'} u_\nu^a}] \quad (19)$$

with  $\mathbf{u}^s = \frac{1}{2}(\mathbf{u} + \tilde{\mathbf{u}})$ ,  $\mathbf{u}^a = \frac{1}{2}(\mathbf{u} - \tilde{\mathbf{u}})$ . One expects

$$(\tilde{\mathbf{u}}, \mathbf{u}) > 0 \quad (20)$$

provided that the "magnitude" of  $\eta_{\dots}^{(a)}$  is "smaller" than  $\eta_{\dots}^{(s)}$ . More specifically, this condition is equivalent to

$$|\operatorname{Im} \eta_m| < \operatorname{Re} \eta_m \quad (20a)$$

where  $\eta_m$  ( $m=0, \pm 1, \pm 2$ ) are the complex viscosity coefficients as introduced in Ref. [6]. Incidentally, the magnetic field dependent viscosity of neutral molecular gases (Senftleben-Beenakker effect) always obeys the inequality (20a). For a gaseous plasma, however, the inequality (20a) can be violated if the magnetic field exceeds a critical value.

## Variational Principle

As before, the solution of Eq. (6) with (3) subject to the appropriate boundary conditions is denoted by  $\mathbf{v}$ . The force and torque acting on an obstacle in the fluid are determined by  $(\mathbf{v}, \mathbf{v})$ , cf. Eqs. (8), (9). According to Eq. (16) (put  $\mathbf{v}' = \mathbf{v}'' = \mathbf{v}$ ), this quantity is also equal to  $(\tilde{\mathbf{v}}, \mathbf{v})$ .

The variational principal can now be stated as follows. The functional  $(\tilde{\mathbf{v}}', \mathbf{v}')$  where  $\mathbf{v}'$  is a velocity field which obeys the boundary conditions is extremal for the true solution  $\mathbf{v}$ , i.e.

$$\delta(\tilde{\mathbf{v}}, \mathbf{v}) = 0. \quad (21)$$

In other words, the rate of entropy production is stationary. To prove this relation (21) write  $\mathbf{v}'$  as  $\mathbf{v}' = \mathbf{v} + \delta\mathbf{v}$ . The orthogonality relations (14) (put  $\delta\mathbf{v}' = \delta\tilde{\mathbf{v}}$ ,  $\mathbf{v}'' = \mathbf{v}$ ) and (15) (put  $\tilde{\mathbf{v}}'' = \tilde{\mathbf{v}}$ ,  $\delta\mathbf{v}' = \delta\mathbf{v}$ ) then imply that terms linear in  $\delta\mathbf{v}$ ,  $\delta\tilde{\mathbf{v}}$  vanish in the difference  $(\tilde{\mathbf{v}}', \mathbf{v}') - (\tilde{\mathbf{v}}, \mathbf{v})$  [8].

## Upper and Lower Bounds

The variational principle (21) holds true for arbitrary anisotropic viscosity tensor. The upper and lower bounds on  $(\tilde{\mathbf{v}}, \mathbf{v})$  and consequently on the force and torque are derived for the case where the inequality (20) holds true, i.e. if the antisymmetric part of  $\eta, \dots$  is smaller than its symmetric part.

Now, the expression  $(\tilde{\mathbf{v}}' - \tilde{\mathbf{v}}'', \mathbf{v}' - \mathbf{v}'')$  is considered where  $\mathbf{v}'$  and  $\mathbf{v}''$  obey the boundary conditions and the differential equation, respectively.

With

$$\begin{aligned} \mathbf{v}' - \mathbf{v}'' &= \mathbf{v}' - \mathbf{v} + \mathbf{v} - \mathbf{v}'', \\ \tilde{\mathbf{v}}' - \tilde{\mathbf{v}}'' &= \tilde{\mathbf{v}}' - \tilde{\mathbf{v}} + \tilde{\mathbf{v}} - \tilde{\mathbf{v}}'', \\ (\tilde{\mathbf{v}}' - \tilde{\mathbf{v}}'', \mathbf{v}' - \mathbf{v}'') &= (\tilde{\mathbf{v}}' - \tilde{\mathbf{v}}, \mathbf{v}' - \mathbf{v}) + (\tilde{\mathbf{v}} - \tilde{\mathbf{v}}'', \mathbf{v} - \mathbf{v}'') \end{aligned} \quad (22)$$

is obtained where use has been made of the orthogonality relations (14), (15). Again,  $\mathbf{v}$  denotes the true solution of the problem.

Subject to the inequality (20), all three scalar products occurring in (22) are positive definite.

### i) Upper Bound

For  $\mathbf{v}'' = 0$ , Eq. (22) reduces to

$$(\tilde{\mathbf{v}}', \mathbf{v}') = (\tilde{\mathbf{v}}' - \tilde{\mathbf{v}}, \mathbf{v}' - \mathbf{v}) + (\tilde{\mathbf{v}}, \mathbf{v}). \quad (23)$$

Provided that the inequality (20) is fulfilled, one has  $(\tilde{\mathbf{v}}' - \tilde{\mathbf{v}}, \mathbf{v}' - \mathbf{v}) > 0$  and (23) leads to the desired

upper bound

$$(\mathbf{v}, \mathbf{v}) = (\tilde{\mathbf{v}}, \mathbf{v}) < (\tilde{\mathbf{v}}', \mathbf{v}'). \quad (24)$$

This inequality expresses the well-known fact [5, 7] that the entropy production has its minimum in a stationary state. Here, this relation has been proved for an anisotropic (and spatially inhomogeneous) fluid. However, it should be stressed again that the inequality (24) can be violated in a fluid with a nonsymmetric viscosity tensor if relation (20) is not obeyed.

### ii) Lower Bound

Subject to the condition stated above, the inequality

$$(\tilde{\mathbf{v}}' - \tilde{\mathbf{v}}'', \mathbf{v}' - \mathbf{v}'') < (\tilde{\mathbf{v}}' - \tilde{\mathbf{v}}, \mathbf{v}' - \mathbf{v}) \quad (25)$$

is inferred from (22). This expression is equivalent to

$$\begin{aligned} (\tilde{\mathbf{v}}', \mathbf{v}') - (\tilde{\mathbf{v}}', \mathbf{v}'') - (\tilde{\mathbf{v}}'', \mathbf{v}') + (\tilde{\mathbf{v}}'', \mathbf{v}'') \\ < (\tilde{\mathbf{v}}', \mathbf{v}') - (\tilde{\mathbf{v}}, \mathbf{v}) - (\tilde{\mathbf{v}}, \mathbf{v}') + (\tilde{\mathbf{v}}, \mathbf{v}). \end{aligned}$$

Due to  $(\tilde{\mathbf{v}}', \mathbf{v}) = (\tilde{\mathbf{v}}, \mathbf{v})$  and  $(\tilde{\mathbf{v}}, \mathbf{v}') = (\tilde{\mathbf{v}}, \mathbf{v})$  cf. Eq. (16),

$$(\tilde{\mathbf{v}}, \mathbf{v}'') + (\tilde{\mathbf{v}}'', \mathbf{v}') - (\tilde{\mathbf{v}}'', \mathbf{v}'') < (\tilde{\mathbf{v}}, \mathbf{v}) \quad (26)$$

is obtained. It is recalled that  $\mathbf{v}''$  is assumed to obey the linear differential equation (6) with (3). The function  $\alpha \mathbf{v}''$  where  $\alpha$  is a numerical factor is also a solution of (3) provided that  $\nabla p$  is scaled by the same factor  $\alpha$ . If  $\alpha \mathbf{v}''$  is used as trial function instead of  $\mathbf{v}''$  in (26), the inequality

$$\alpha[(\tilde{\mathbf{v}}, \mathbf{v}'') + (\tilde{\mathbf{v}}'', \mathbf{v}')] - \alpha^2(\tilde{\mathbf{v}}'', \mathbf{v}'') < (\tilde{\mathbf{v}}, \mathbf{v}) \quad (27)$$

is found. The best lower bound is obtained for the value of  $\alpha$  which maximizes the left hand side of (27), viz.

$$\alpha = \frac{1}{2}[(\tilde{\mathbf{v}}, \mathbf{v}'') + (\tilde{\mathbf{v}}'', \mathbf{v}')] (\tilde{\mathbf{v}}'', \mathbf{v}'')^{-1}. \quad (28)$$

Insertion of (28) into (27) finally yields the desired lower bound

$$\frac{1}{4} \frac{[(\tilde{\mathbf{v}}, \mathbf{v}'') + (\tilde{\mathbf{v}}'', \mathbf{v}')]^2}{(\tilde{\mathbf{v}}'', \mathbf{v}'')} < (\tilde{\mathbf{v}}, \mathbf{v}) = (\mathbf{v}, \mathbf{v}). \quad (29)$$

This inequality is not affected if  $\mathbf{v}''$  is multiplied by some finite factor. For a fluid with a symmetric viscosity tensor relation (29) reduces to

$$(\mathbf{v}', \mathbf{v}')^2 (\mathbf{v}'', \mathbf{v}'')^{-1} < (\mathbf{v}, \mathbf{v}). \quad (30)$$

This inequality is formally equivalent to a similar expression which has been derived for elastic solid [1, 2].



### Concluding Remarks

In this article, a variational principle for the rate of entropy production has been derived for a fluid with anisotropic and spatially inhomogeneous viscosity. Subject to condition (20), upper and lower bounds have been established for this quantity, cf. Eqs. (24), (29). Thus the frictional force and torque exerted by a fluid on solid particles can be bounded from above and below. The derivation of these bounds is based on the linearized Navier-Stokes equation and it is assumed that the viscosity tensor is independent of the velocity gradient (Newtonian viscosity). An extension of the present method to

the case of a non-Newtonian viscosity would be desirable.

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